



# Another Coboundary Operator for Differential Forms with Values in the Lie Algebra Bundle of a Group Bundle -A Chapter in Synthetic Differential Geometry of Groupoids-

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**ANOTHER COBOUNDARY OPERATOR FOR  
DIFFERENTIAL FORMS WITH VALUES IN THE LIE  
ALGEBRA OF A GROUP  $\triangleleft$  A CHAPTER IN  
SYNTHETIC DIFFERENTIAL GEOMETRY OF  
GROUPOIDS  $\triangleright$**

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**Abstract**

Kock [5] has considered differential forms with values in a group in a context where neighborhood relations are available. By doing so, he has made it clear where the so-called Maurer-Cartan formula should come from. In this paper, while we retain the classical definition of differential form with values in the Lie algebra of a group, we propose another definition of coboundary operator for the de Rham complex in a highly general microlinear context, in which neighborhood relations are no longer in view. Using this new definition of the coboundary operator, it is to be shown that the main result of Kock's paper mentioned above still prevails in our general microlinear context. Our considerations will be carried out within the framework of groupoids.

**1. Introduction**

Although the Maurer-Cartan equation has long been known, it is

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Kock [5] that excavated its hidden geometric meaning for the first time. To this end, he introduced differential forms with values in groups in place of classical ones with values in their Lie algebras by using first neighborhood relations. We do not know exactly what spaces enjoy first neighborhood relations, but we are sure at least that formal manifolds are naturally endowed with such relations. As far as formal manifolds are concerned, he has demonstrated that his new definition of differential form is anyway equivalent to the classical one. This means that if we want to generalize his geometric ideas from formal manifolds to microlinear spaces in general, it is not necessary to adhere to his noble definition of differential form. In other words, we can say that his novel definition of differential form does not constitute the indispensable components of his thrilling geometric ideas.

Another unsatisfactory feature of [5] is that his proof on the exact discrepancy between the coboundary operator of the de Rham complex and his contour derivative from dimension 1 to dimension 2, from which the Maurer-Cartan equation comes at once, appears rather analytic than synthetic, though he is very famous in synthetic differential geometry.

We are inexpiably diehard in the definition of differential form, so that we prefer differential forms with values in their Lie algebras to ones with values in groups themselves. We prefer cubical arguments to simplicial ones which Kock admires. However we propose another definition for the coboundary operator of the de Rham complex, which is to be eventually shown to be equivalent to the classical one in synthetic differential geometry. This new definition of the coboundary operator facilitates the comparison between the coboundary operator of the de Rham complex and the cubical version of his contour derivative from dimension 1 to dimension 2. Thus this paper might be put down as a microlinear generalization of Kock's [5] ingenious ideas.

The paper is organized as follows. After giving some preliminaries and fixing our notation in the succeeding section, we will deal with the coboundary operator for the de Rham complex in a somewhat general context, in which differential forms on a groupoid  $G$  over a base space  $M$  with values in a vector bundle  $E$  over the same base space  $M$  are considered in Section 3. The coboundary operator studied here is called

the *additive coboundary operator* and is denoted by  $\mathbf{d}_+$ . If the vector bundle  $E$  happens to be the Lie algebra bundle  $\mathcal{A}L$  of a group bundle  $L$  over  $M$ , another definition of the coboundary operator naturally emerges besides the additive one with due regard to group structures that  $L$  possesses. The emerging coboundary operator is called the *multiplicative coboundary operator* and is denoted by  $\mathbf{d}_\times$ . This is the topic of Section 4. The succeeding section is devoted to establishing the coincidence of  $\mathbf{d}_+$  and  $\mathbf{d}_\times$  whenever both are available. The last section is devoted to a microlinear generalization of Kock's [5] main result. Our standard reference on synthetic differential geometry is Lavendhomme [7]. Unless stated to the contrary, every space in this paper is assumed to be microlinear. Our discussions will be carried out within the context of groupoids, which is a bit more general than Kock's one.

## 2. Preliminaries

### 2.1. Groupoids

Given  $x \in M$  and a groupoid  $G$  over a base  $M$  with its object inclusion map  $\text{id} : M \rightarrow G$  and its source and target projections  $\alpha, \beta : G \rightarrow M$ , we denote by  $\mathcal{A}_x^n G$  the totality of mappings  $\gamma : D^n \rightarrow G$  with  $\gamma(0, \dots, 0) = \text{id}_x$  and  $(\alpha \circ \gamma)(d_1, \dots, d_n) = x$  for any  $(d_1, \dots, d_n) \in D^n$ . We denote by  $\mathcal{A}^n G$  the set-theoretic union of  $\mathcal{A}_x^n G$ 's for all  $x \in M$ . In particular, we often write  $\mathcal{A}_x G$  and  $\mathcal{A}G$  in place of  $\mathcal{A}_x^1 G$  and  $\mathcal{A}^1 G$ . Given  $t \in \mathcal{A}G$  and  $d \in D$ , we usually write  $t_d$  instead of  $t(d)$ . By the same token as in Propositions 1 and 2 of [7, Section 3.1], it is easy to see that  $\mathcal{A}G$  can naturally be regarded as a vector bundle over  $M$  (i.e.,  $\mathcal{A}_x G$  is a Euclidean  $\mathbb{R}$ -module for any  $x \in M$ , where  $\mathbb{R}$  stands for the set of real numbers with a cornucopia of nilpotent infinitesimals pursuant to the general Kock-Lawvere axiom).

Given a group bundle  $L$  over  $M$ ,  $\mathcal{A}L$  is not only a vector bundle over  $M$  but also a Lie algebra bundle over  $M$ , where the Lie bracket  $[\cdot, \cdot]$  is defined by the following proposition.

**Proposition 1.** *Let  $x \in M$ . Given  $t_1, t_2 \in \mathcal{A}_x L$ , there exists a unique  $[t_1, t_2] \in \mathcal{A}_x L$  such that*

$$[t_1, t_2]_{d_1 d_2} = (t_2)_{-d_2} (t_1)_{-d_1} (t_2)_{d_2} (t_1)_{d_1} \quad (1)$$

for any  $d_1, d_2 \in D$ .

**Proof.** By Proposition 7 of [7, Section 2.2], it suffices to note that if  $d_1 = 0$  or  $d_2 = 0$ , then the right hand side of (1) is equal to  $\text{id}_x$ , which is easy to see.  $\square$

**Theorem 2.** *With respect to  $[\cdot, \cdot]$  defined above,  $\mathcal{A}_x L$  is a Lie algebra.*

**Proof.** By the same token as in our [11].  $\square$

We note the following simple proposition in passing.

**Proposition 3.** *Let  $x \in M$ . Given  $t_1, t_2 \in \mathcal{A}_x L$ , we have*

$$(t_1 + t_2)_d = (t_2)_d (t_1)_d = (t_1)_d (t_2)_d$$

for any  $d \in D$ .

**Proof.** By the same token as in Proposition 7 of [11].  $\square$

Given a vector bundle  $E$  over  $M$ , we denote by  $\Phi_{\text{Lin}}(E)$  the linear frame groupoid of  $E$ . A morphism of groupoids from a groupoid  $G$  to  $\Phi_{\text{Lin}}(E)$  is called a *representation* of  $G$  on  $E$ . Given a group bundle  $L$  over  $M$ , we denote by  $\Phi_{\text{Grp}}(L)$  the group frame groupoid of  $L$ . A morphism of groupoids from a groupoid  $G$  to  $\Phi_{\text{Grp}}(L)$  is called a *representation* of  $G$  on  $L$ . Given a representation  $\rho$  of  $G$  on  $L$ , it gives rise naturally to a representation  $\rho_{\text{Lie}}$  of  $G$  on the vector bundle  $\mathcal{A}L$ . Given  $f : x \rightarrow y$  in  $G$  and  $t \in \mathcal{A}_x L$ ,  $\rho_{\text{Lie}}(f)(t)$  is defined to be

$$\rho_{\text{Lie}}(f)(t)_d = \rho(f)(t_d)$$

for any  $d \in D$ . It is easy to see that  $\rho_{\text{Lie}}$  is not only a morphism of groupoids from  $G$  to  $\Phi_{\text{Lin}}(\mathcal{A}L)$  but also a morphism of groupoids from  $G$

to  $\Phi_{\text{Lie}}(\mathcal{A}L)$ , where  $\Phi_{\text{Lie}}(\mathcal{A}L)$  denotes the Lie algebra frame groupoid of  $\mathcal{A}L$ .

Our standard reference on groupoids is [8].

## 2.2. Differential Forms

Given a groupoid  $G$  and a vector bundle  $E$  over the same space  $M$ , the space  $\mathbf{C}^n(G, E)$  of *differential  $n$ -forms with values in  $E$*  consists of all mappings  $\omega$  from  $\mathcal{A}^n G$  to  $E$  whose restriction to  $\mathcal{A}_x^n G$  for each  $x \in M$  takes values in  $E_x$  satisfying the following  $n$ -homogeneous and alternating properties:

1. We have

$$\omega(a \underset{i}{:} \gamma) = a\omega(\gamma) \quad (1 \leq i \leq n)$$

for any  $a \in \mathbb{R}$  and any  $\gamma \in \mathcal{A}_x^n G$ , where  $a \underset{i}{:} \gamma \in \mathcal{A}_x^n G$  is defined to be

$$(a \underset{i}{:} \gamma)(d_1, \dots, d_n) = \gamma(d_1, \dots, d_{i-1}, ad_i, d_{i+1}, \dots, d_n)$$

for any  $(d_1, \dots, d_n) \in D^n$ .

2. We have

$$\omega(\gamma \circ D^\sigma) = \text{sign}(\sigma)\omega(\gamma)$$

for any permutation  $\sigma$  of  $\{1, \dots, n\}$ , where  $D^\sigma : D^n \rightarrow D^n$  permutes the  $n$  coordinates by  $\sigma$ .

## 2.3. Two Infinitesimal Stokes' Theorems

Let  $E$  be a vector bundle over  $M$ . If  $\omega \in \mathbf{C}^n(G, E)$ , then the mapping  $\varphi_\omega : \mathcal{A}^n G \times D^n \rightarrow E$  defined by

$$\varphi_\omega(\gamma, d_1, \dots, d_n) = d_1 \cdots d_n \omega(\gamma)$$

abides by the following conditions:

1. We have

$$\varphi_{\omega}(a \underset{i}{:} \gamma, d_1, \dots, d_n) = a\varphi_{\omega}(\gamma, d_1, \dots, d_n) \quad (1 \leq i \leq n)$$

for any  $a \in \mathbb{R}$ .

2. We have

$$\begin{aligned} & \varphi_{\omega}(\gamma, d_1, \dots, d_{i-1}, ad_i, d_{i+1}, \dots, d_n) \\ &= a\varphi_{\omega}(\gamma, d_1, \dots, d_{i-1}, d_i, d_{i+1}, \dots, d_n) \quad (1 \leq i \leq n) \end{aligned}$$

for any  $a \in \mathbb{R}$ .

3. We have

$$\varphi_{\omega}(\gamma \circ D^{\sigma}, d_1, \dots, d_n) = \text{sign}(\sigma)\varphi_{\omega}(\gamma, d_1, \dots, d_n)$$

for any permutation  $\sigma$  of  $\{1, \dots, n\}$ .

Conversely we have.

**Theorem 4.** *If  $\varphi: \mathcal{A}^n G \times D^n \rightarrow E$  satisfies the above three conditions, then there exists a unique  $\omega_{\varphi} \in \mathbf{C}^n(G, E)$  such that*

$$\varphi(\gamma, d_1, \dots, d_n) = d_1 \cdots d_n \omega_{\varphi}(\gamma)$$

for any  $\gamma \in \mathcal{A}^n G$  and any  $(d_1, \dots, d_n) \in D^n$ .

**Proof.** By the same token as in the proof of Proposition 2 of Section 4.2 of Lavendhomme [7].  $\square$

Let  $L$  be a group bundle over  $M$ . If  $\omega \in \mathbf{C}^n(G, \mathcal{A}L)$ , then the mapping  $\varphi_{\omega}: \mathcal{A}^n G \times D^n \rightarrow L$  defined by

$$\varphi_{\omega}(\gamma, d_1, \dots, d_n) = \omega(\gamma)(d_1 \cdots d_n)$$

abides by the following conditions:

1. We have

$$\varphi_{\omega}(a \underset{i}{:} \gamma, d_1, \dots, d_n) = \varphi_{\omega}(\gamma, ad_1, d_2, \dots, d_n) \quad (1 \leq i \leq n)$$

for any  $a \in \mathbb{R}$ .

2. We have

$$\begin{aligned} & \varphi_\omega(\gamma, d_1, \dots, d_{i-1}, ad_i, d_{i+1}, \dots, d_{j-1}, d_j, d_{j+1}, \dots, d_n) \\ &= \varphi_\omega(\gamma, d_1, \dots, d_{i-1}, d_i, d_{i+1}, \dots, d_{j-1}, ad_j, d_{j+1}, \dots, d_n) \quad (1 \leq i < j \leq n) \end{aligned}$$

for any  $a \in \mathbb{R}$ .

3. We have

$$\varphi_\omega(\gamma \circ D^\sigma, d_1, \dots, d_n) = \varphi_\omega(\gamma, d_1, \dots, d_n)^{\text{sign}(\sigma)}$$

for any permutation  $\sigma$  of  $\{1, \dots, n\}$ .

Conversely we have

**Theorem 5.** *If  $\varphi: \mathcal{A}^n G \times D^n \rightarrow L$  satisfies the above three conditions, then there exists a unique  $\omega_\varphi \in \mathbf{C}^n(G, AL)$  such that*

$$\varphi(\gamma, d_1, \dots, d_n) = \omega_\varphi(\gamma)(d_1 \cdots d_n)$$

for any  $\gamma \in \mathcal{A}^n G$  and any  $(d_1, \dots, d_n) \in D^n$ .

**Proof.** By the same token as in the proof of Proposition 2 of Section 4.2 of Lavendhomme [7] except for the following quasi-colimit diagram of small objects in place of the corresponding one there.  $\square$

**Lemma 6.** *The diagram*

$$\begin{array}{ccc} & \tau_1 & \\ & \rightarrow & \\ D^{n+1} & \vdots & D^n \xrightarrow{\mathbf{m}} D \\ & \tau_n & \\ & \rightarrow & \end{array}$$

is a quasi-colimit diagram of small objects, where

$$\tau_i(d_0, d_1, \dots, d_n) = (d_1, \dots, d_{i-1}, d_0 d_i, d_{i+1}, \dots, d_n) \quad (1 \leq i \leq n)$$

for any  $(d_0, d_1, \dots, d_n) \in D^{n+1}$ , and

$$\mathbf{m}(d_1, \dots, d_n) = d_1 \cdots d_n$$

for any  $(d_1, \dots, d_n) \in D^n$ .



### 3. The Additive Complex

Let  $\rho : G \rightarrow \Phi_{\text{Lin}}(E)$  be a representation of the groupoid  $G$  on a vector bundle  $E$ .<sup>1</sup> These entities shall be fixed throughout this section. Given  $\gamma \in \mathcal{A}^{n+1}G$  and  $e \in D$ , we define  $\gamma_e^i \in \mathcal{A}^n G$  ( $1 \leq i \leq n+1$ ) to be

$$\gamma_e^i(d_1, \dots, d_n) = \gamma(d_1, \dots, d_{i-1}, e, d_i, \dots, d_n) \gamma(0, \dots, 0, e, 0, \dots, 0)^{-1}$$

for any  $(d_1, \dots, d_n) \in D^n$ . Similarly, given  $\gamma \in \mathcal{A}^{n+1}G$ , we define  $\gamma_i \in \mathcal{A}G$  ( $1 \leq i \leq n+1$ ) to be

$$\gamma_i(d) = \gamma(0, \dots, 0, d, 0, \dots, 0)$$

for any  $d \in D$ .

**Theorem 7.** *Given  $\omega \in \mathbf{C}^n(G, \mathcal{A}L)$ , there exists a unique  $\mathbf{d}_+\omega \in \mathbf{C}^{n+1}(G, \mathcal{A}L)$  such that*

$$\begin{aligned} & d_1 \cdots d_{n+1} \mathbf{d}_+\omega(\gamma) \\ &= \sum_{i=1}^{n+1} (-1)^i d_1 \cdots \hat{d}_i \cdots d_{n+1} \{ \omega(\gamma_0^i) - \rho_{((\gamma_i)_{d_i})^{-1}}(\omega(\gamma_{d_i}^i)) \} \end{aligned} \quad (2)$$

for any  $\gamma \in \mathcal{A}^{n+1}G$  and any  $(d_1, \dots, d_{n+1}) \in D^{n+1}$ .

**Proof.** By Theorem 4, it suffices to note that the function  $\varphi : \mathcal{A}^{n+1}G \times D^{n+1} \rightarrow L$  defined by

$$\begin{aligned} & \varphi(\gamma, d_1, \dots, d_{n+1}) \\ &= \sum_{i=1}^{n+1} (-1)^i d_1 \cdots \hat{d}_i \cdots d_{n+1} \{ \omega(\gamma_0^i) - \rho_{((\gamma_i)_{d_i})^{-1}}(\omega(\gamma_{d_i}^i)) \} \end{aligned}$$

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<sup>1</sup> Notes added in proof: As was pointed out by Professor Anders Kock (Aarhus University), it is too much to assume a representation  $\rho : G \rightarrow \Phi_{\text{Grp}}(L)$  in Sections 3-5. It should be replaced by a representation of  $\sigma : \mathcal{A}G \rightarrow \mathcal{A}(\Phi_{\text{Lin}}(L))$ , i.e., a homomorphism of Nishimura algebroids from  $\mathcal{A}G$  to  $\mathcal{A}(\Phi_{\text{Lin}}(L))$ , the notation and terminology of which the reader is referred to “H. Nishimura, A functional approach to the infinitesimal theory of groupoids, Math. ArXiv 0709 3629 (Sep. 2007).”

for any  $\gamma \in \mathcal{A}^{n+1}G$  and any  $(d_1, \dots, d_{n+1}) \in D^{n+1}$  satisfies the three conditions mentioned therein. We fix a notation in passing. Let  $F_i : D \rightarrow E$  be the assignment of  $\rho_{((\gamma_i)_d)^{-1}}(\omega(\gamma_d^i)) \in E$  to each  $d \in D$ , for which we have

$$\omega(\gamma_0^i) - \rho_{((\gamma_i)_d)^{-1}}(\omega(\gamma_d^i)) = -d\mathbf{D}F_i,$$

where  $\mathbf{D}F_i$  is the derivative of  $F_i$  at 0.

1. For the first condition, we have to show that

$$\varphi(a \underset{j}{\cdot} \gamma, d_1, \dots, d_n) = \alpha \varphi(\gamma, d_1, \dots, d_n) \quad (1 \leq j \leq n)$$

for any  $a \in \mathbb{R}$ . If  $i < j$ , then we have

$$\begin{aligned} & \omega((a \underset{j}{\cdot} \gamma)_0^i) - \rho_{(((a \underset{j}{\cdot} \gamma)_i)_{d_i})^{-1}}(\omega((a \underset{j}{\cdot} \gamma)_{d_i}^i)) \\ &= \omega((a \underset{j-1}{\cdot} (\gamma_0^i))) - \rho_{((\gamma_i)_{d_i})^{-1}}(\omega((a \underset{j-1}{\cdot} (\gamma_{d_i}^i)))) \\ &= \alpha \{\omega(\gamma_0^i) - \rho_{((\gamma_i)_{d_i})^{-1}}(\omega(\gamma_{d_i}^i))\}. \end{aligned}$$

If  $j < i$ , then we have

$$\begin{aligned} & \omega((a \underset{j}{\cdot} \gamma)_0^i) - \rho_{(((a \underset{j}{\cdot} \gamma)_i)_{d_i})^{-1}}(\omega((a \underset{j}{\cdot} \gamma)_{d_i}^i)) \\ &= \omega((a \underset{j}{\cdot} (\gamma_0^i))) - \rho_{((\gamma_i)_{d_i})^{-1}}(\omega((a \underset{j}{\cdot} (\gamma_{d_i}^i)))) \\ &= \alpha \{\omega(\gamma_0^i) - \rho_{((\gamma_i)_{d_i})^{-1}}(\omega(\gamma_{d_i}^i))\}. \end{aligned}$$

Finally we consider the case of  $j = i$ , in which we have

$$\begin{aligned} & \omega((a \underset{i}{\cdot} \gamma)_0^i) - \rho_{(((a \underset{i}{\cdot} \gamma)_i)_{d_i})^{-1}}(\omega((a \underset{i}{\cdot} \gamma)_{d_i}^i)) \\ &= \omega(\gamma_0^i) - \rho_{((\gamma_i)_{ad_i})^{-1}}(\omega(\gamma_{ad_i}^i)) \end{aligned}$$

$$\begin{aligned}
&= -ad_i \mathbf{D}F_i \\
&= a\{\omega(\gamma_0^i) - \rho_{((\gamma_i)_{d_i})^{-1}}(\omega(\gamma_{d_i}^i))\}.
\end{aligned}$$

2. The easy verification of the second condition is left to the reader.

3. For the third condition, it suffices to note that

$$\begin{aligned}
&d_1 \cdots \hat{d}_i \cdots d_{n+1} \{\omega(\gamma_0^i) - \rho_{((\gamma_i)_{d_i})^{-1}}(\omega(\gamma_{d_i}^i))\} \\
&= -d_1 \cdots d_{n+1} \mathbf{D}F_i \\
&= d_1 \cdots \hat{d}_j \cdots d_{n+1} \{\omega(\gamma_0^i) - \rho_{((\gamma_i)_{d_j})^{-1}}(\omega(\gamma_{d_j}^i))\}
\end{aligned}$$

for any  $i \neq j$ . □

Now we would like to show that  $\mathbf{d}_+^2 = 0$ , for which we need three lemmas. Let  $\gamma \in \mathcal{A}^{n+2}G$  throughout the following three lemmas.

**Lemma 8.** *For  $1 \leq j \leq i \leq n+1$ , we have*

$$(\gamma_0^j)_i = \gamma_{i+1}.$$

*For  $1 \leq i < j \leq n+2$ , we have*

$$(\gamma_0^j)_i = \gamma_i.$$

**Proof.** Obvious. □

**Lemma 9.** *For  $1 \leq j < i \leq n+2$  and  $e, e' \in D$ , we have*

$$(\gamma_e^i)_{e'}^j = (\gamma_{e'}^j)_e^{i-1}.$$

**Proof.** It is easy to see that both  $(\gamma_e^i)_{e'}^j$  and  $(\gamma_{e'}^j)_e^{i-1}$  are the same mapping

$$\begin{aligned}
&(d_1, \dots, d_{n+2}) \in D^{n+2} \\
&\mapsto \gamma(d_1, \dots, d_{j-1}, e', d_j, \dots, d_{i-1}, e, d_i, \dots, d_n) \\
&\gamma(0, \dots, 0, e'_j, 0, \dots, 0, e'_i, 0, \dots, 0)^{-1}.
\end{aligned}$$
□

**Lemma 10.** For  $1 \leq j < i \leq n+2$  and  $d_i, d_j \in D$ , we have

$$\rho_{((\gamma_i)_{d_i})^{-1}} \circ \rho_{(((\gamma_{d_i}^i)_j)_{d_j})^{-1}} = \rho_{((\gamma_j)_{d_j})^{-1}} \circ \rho_{(((\gamma_{d_j}^j)_{i-1})_{d_i})^{-1}}.$$

**Proof.** It is easy to see that both  $\rho_{((\gamma_i)_{d_i})^{-1}} \circ \rho_{(((\gamma_{d_i}^i)_j)_{d_j})^{-1}}$  and  $\rho_{((\gamma_j)_{d_j})^{-1}} \circ \rho_{(((\gamma_{d_j}^j)_{i-1})_{d_i})^{-1}}$  are equal to  $\rho_{\gamma(0, \dots, 0, \underset{j}{1}, 0, \dots, 0, \underset{i}{2}, 0, \dots, 0)^{-1}}$ .  $\square$

**Theorem 11.** We have

$$\mathbf{d}_+^2 = 0.$$

In other words, the composition

$$\mathbf{C}^n(G, E) \xrightarrow{\mathbf{d}_+} \mathbf{C}^{n+1}(G, E) \xrightarrow{\mathbf{d}_+} \mathbf{C}^{n+2}(G, E)$$

vanishes.

**Proof.** Let  $\gamma \in \mathcal{A}^{n+2}G$ ,  $d_1, \dots, d_{n+2} \in D$  and  $\omega \in \mathbf{C}^n(G, E)$ . Then we have

$$\begin{aligned} & d_1 \cdots d_{n+2} \mathbf{d}_+^2 \omega(\gamma) \\ &= \sum_{i=1}^{n+2} (-1)^i d_1 \cdots \hat{d}_i \cdots d_{n+2} \mathbf{d}_+ \omega(\gamma_0^i) \\ &\quad - \sum_{i=1}^{n+2} (-1)^i d_1 \cdots \hat{d}_i \cdots d_{n+2} \rho_{\gamma_i(d_i)^{-1}} (\mathbf{d}_+ \omega(\gamma_{d_i}^i)) \\ &= \sum_{i=1}^{n+2} \sum_{j=1}^{i-1} (-1)^{i+j} d_1 \cdots \hat{d}_j \cdots \hat{d}_i \cdots d_{n+2} \omega((\gamma_0^i)_0^j) \\ &\quad + \sum_{i=1}^{n+2} \sum_{j=1}^{i-1} (-1)^{i+j+1} d_1 \cdots \hat{d}_j \cdots \hat{d}_i \cdots d_{n+2} \rho_{(\gamma_0^i)_j(d_j)^{-1}} (\omega((\gamma_0^i)_{d_j}^j)) \\ &\quad + \sum_{i=1}^{n+2} \sum_{j=1}^{i-1} (-1)^{i+j+1} d_1 \cdots \hat{d}_j \cdots \hat{d}_i \cdots d_{n+2} \rho_{\gamma_i(d_i)^{-1}} (\omega((\gamma_{d_i}^i)_0^j)) \\ &\quad + \sum_{i=1}^{n+2} \sum_{j=1}^{i-1} (-1)^{i+j} d_1 \cdots \hat{d}_j \cdots \hat{d}_i \cdots d_{n+2} \rho_{\gamma_i(d_i)^{-1}} \circ \rho_{(\gamma_{d_i}^i)_j(d_j)^{-1}} (\omega((\gamma_{d_i}^i)_{d_j}^j)) \end{aligned}$$

$$\begin{aligned}
& + \sum_{i=1}^{n+2} \sum_{j=i}^{n+1} (-1)^{i+j} d_1 \cdots \hat{d}_i \cdots \widehat{d_{j+1}} \cdots d_{n+2} \omega((\gamma_0^i)_0^j) \\
& + \sum_{i=1}^{n+2} \sum_{j=i}^{n+1} (-1)^{i+j+1} d_1 \cdots \hat{d}_i \cdots \widehat{d_{j+1}} \cdots d_{n+2} \rho_{(\gamma_0^i)_j(d_j)^{-1}}(\omega((\gamma_0^i)_{d_j}^j)) \\
& + \sum_{i=1}^{n+2} \sum_{j=i}^{n+1} (-1)^{i+j+1} d_1 \cdots \hat{d}_i \cdots \widehat{d_{j+1}} \cdots d_{n+2} \rho_{\gamma_i(d_i)^{-1}}(\omega((\gamma_{d_i}^i)_0^j)) \\
& + \sum_{i=1}^{n+2} \sum_{j=i}^{n+1} (-1)^{i+j} d_1 \cdots \hat{d}_i \cdots \widehat{d_{j+1}} \cdots d_{n+2} \rho_{\gamma_i(d_i)^{-1}} \\
& \quad \circ \rho_{(\gamma_{d_i}^i)_j(d_j)^{-1}}(\omega((\gamma_{d_i}^i)_{d_j}^j)) \\
& = 0.
\end{aligned}$$

The final derivation of total vanishment comes from the cancellation of the first double summation and the fifth one in the previous development by Lemma 9, that of the second double summation and the seventh one in the previous development by Lemmas 7 and 9, that of the third double summation and the sixth one in the previous development by Lemmas 7 and 9, and finally that of the fourth double summation and the eighth one in the previous development by Lemmas 9 and 10.  $\square$

The operator  $\mathbf{d}_+$  is called the *additive coboundary operator with respect to the representation  $\rho$* .

#### 4. The Multiplicative Complex

Let  $\rho : G \rightarrow \Phi_{\text{Grp}}(L)$  be a representation of the groupoid  $G$  on a group bundle  $L$ , which naturally gives rise to an induced representation  $\rho_{\text{Lie}} : G \rightarrow \Phi_{\text{Lie}}(\mathcal{A}L)$ .<sup>1</sup> Therefore we have the additive coboundary operator

$$\mathbf{d}_+ : \mathbf{C}^n(G, \mathcal{A}L) \rightarrow \mathbf{C}^{n+1}(G, \mathcal{A}L)$$

with respect to the induced representation  $\rho_{\text{Lie}}$ , as we have discussed in the preceding section. Now we are going to define another coboundary operator

$$\mathbf{d}_\times : \mathbf{C}^n(G, \mathcal{A}L) \rightarrow \mathbf{C}^{n+1}(G, \mathcal{A}L)$$

to be called the *multiplicative coboundary operator with respect to  $\rho$* . Given  $\omega \in \mathbf{C}^n(G, \mathcal{A}L)$ ,  $\mathbf{d}_\times \omega \in \mathbf{C}^{n+1}(G, \mathcal{A}L)$  is expected to be defined in such a way that

$$\begin{aligned} & ((\mathbf{d}_\times \omega)(\gamma))_{d_1 \dots d_{n+1}} \\ &= \prod_{i=1}^{n+1} \{(\omega(\gamma_0^i))_{d_1 \dots \hat{d}_i \dots d_{n+1}} \rho_{((\gamma_i)_{d_i})^{-1}}((\omega(\gamma_{d_i}^i))_{-d_1 \dots \hat{d}_i \dots d_{n+1}})\}^{(-1)^i} \end{aligned} \quad (3)$$

for any  $\gamma \in \mathcal{A}^{n+1}G$  and any  $(d_1, \dots, d_{n+1}) \in D^{n+1}$ . To show its existence and uniqueness, we need a simple lemma.

**Lemma 12.** *Let  $F_i : D \rightarrow \mathcal{A}L$  be the assignment in the proof of Theorem 7 with  $\mathcal{A}L$  in place of  $E$ . Then we have*

$$\begin{aligned} & (\omega(\gamma_0^i))_{-d} \rho_{((\gamma_i)_e)^{-1}}((\omega(\gamma_e^i))_d) \\ &= (e\mathbf{D}F_i)_d \\ &= (\mathbf{D}F_i)_{de}. \end{aligned}$$

**Proof.** By dint of Proposition 3, the statement is merely a reformulation of

$$F_i(e) = F_i(0) + e\mathbf{D}F_i. \quad \square$$

**Corollary 13.** *The order of multiplication of  $n+1$  factors in (3) does not matter.*

**Proof.** By dint of Proposition 3, it suffices to note that

$$\begin{aligned} & \{(\omega(\gamma_0^i))_{d_1 \dots \hat{d}_i \dots d_{n+1}} \rho_{((\gamma_i)_{d_i})^{-1}}((\omega(\gamma_{d_i}^i))_{-d_1 \dots \hat{d}_i \dots d_{n+1}})\}^{(-1)^i} \\ &= (\mathbf{D}F_i)_{(-1)^{i+1}d_1 \dots d_{n+1}}. \end{aligned} \quad \square$$

**Theorem 14.** *For any  $\omega \in \mathbf{C}^n(G, \mathcal{A}L)$ , there exists a unique  $\mathbf{d}_\times \omega \in \mathbf{C}^{n+1}(G, \mathcal{A}L)$  abiding by condition 3.*

**Proof.** By Theorem 5, it suffices to note that the function  $\varphi : \mathcal{A}^{n+1}G \times D^{n+1} \rightarrow L$  defined by

$$\begin{aligned} & \varphi(\gamma, d_1, \dots, d_{n+1}) \\ &= \prod_{i=1}^{n+1} \{(\omega(\gamma_0^i))_{d_1 \dots \widehat{d_i} \dots d_{n+1}} \rho_{((\gamma_i)_{d_i})^{-1}} ((\omega(\gamma_{d_i}^i))_{-d_1 \dots \widehat{d_i} \dots d_{n+1}})\}^{(-1)^i} \end{aligned}$$

for any  $\gamma \in \mathcal{A}^{n+1}G$  and any  $(d_1, \dots, d_{n+1}) \in D^{n+1}$  satisfies the three conditions mentioned therein. The proof can be carried out as in the proof of Theorem 8 by dint of Lemma 12.  $\square$

It is to be shown in the succeeding section that  $\mathbf{d}_\times^2 = 0$ .

### 5. The Coincidence of the Two Complexes

Let  $\rho : G \rightarrow \Phi_{\text{Grp}}(L)$  be a representation of the groupoid  $G$  on a group bundle  $L$ .<sup>1</sup> Now we come to show the main result of this paper.

**Theorem 15** (The Coincidence Theorem). *The additive coboundary operator  $\mathbf{d}_+ : \mathbf{C}^n(G, \mathcal{A}L) \rightarrow \mathbf{C}^{n+1}(G, \mathcal{A}L)$  with respect to the representation  $\rho_{\text{Lie}} : G \rightarrow \Phi_{\text{Lie}}(\mathcal{A}L)$  and the multiplicative coboundary operator  $\mathbf{d}_\times : \mathbf{C}^n(G, \mathcal{A}L) \rightarrow \mathbf{C}^{n+1}(G, \mathcal{A}L)$  with respect to the representation  $\rho : G \rightarrow \Phi_{\text{Grp}}(L)$  coincide for any natural number  $n$ .*

**Proof.** Let  $\gamma \in \mathcal{A}^{n+1}G$  and  $e_1, \dots, e_{n+1}, d_1, \dots, d_{n+1} \in D$ . We note that

$$\begin{aligned} & (e_1 \cdots e_{n+1}(\mathbf{d}_\times \omega(\gamma)))_{d_1 \dots d_{n+1}} \\ &= (\mathbf{d}_\times \omega(\gamma))_{d_1 e_1 \dots d_{n+1} e_{n+1}} \\ &= \prod_{i=1}^{n+1} ((\omega(\gamma_0^i))_{d_1 e_1 \dots \widehat{d_i e_i} \dots d_{n+1} e_{n+1}} \rho_{((\gamma_i)_{d_i})^{-1}} ((\omega(\gamma_{d_i e_i}^i))_{-d_1 e_1 \dots \widehat{d_i e_i} \dots d_{n+1} e_{n+1}}))^{(-1)^i} \\ &= \prod_{i=1}^{n+1} ((d_i e_i \mathbf{D}F_i)_{-d_1 e_1 \dots \widehat{d_i e_i} \dots d_{n+1} e_{n+1}})^{(-1)^i} \end{aligned}$$

$$\begin{aligned}
&= \prod_{i=1}^{n+1} ((\mathbf{D}F_i)_{-d_1 e_1 \cdots d_{n+1} e_{n+1}})^{(-1)^i} \\
&= \prod_{i=1}^{n+1} ((e_1 \cdots e_{n+1} \mathbf{D}F_i)_{-d_1 \cdots d_{n+1}})^{(-1)^i} \\
&= \left\{ \sum_{i=1}^{n+1} (-1)^i (e_1 \cdots \hat{e}_i \cdots e_{n+1} \omega(\gamma_0^i) \right. \\
&\quad \left. - e_1 \cdots \hat{e}_i \cdots e_{n+1} \rho_{((\gamma_i)_{e_i})^{-1}}(\omega(\gamma_{e_i}^i))) \right\}_{d_1 \cdots d_{n+1}} \\
&= (e_1 \cdots e_{n+1} (\mathbf{d}_+ \omega(\gamma)))_{d_1 \cdots d_{n+1}}.
\end{aligned}$$

Since  $e_1, \dots, e_{n+1}, d_1, \dots, d_{n+1} \in D$  were arbitrary, we can conclude that  $\mathbf{d}_+ \omega(\gamma) = \mathbf{d}_\times \omega(\gamma)$ .  $\square$

**Corollary 16.** *We have*

$$\mathbf{d}_\times^2 = 0.$$

In other words, *the composition*

$$\mathbf{C}^n(G, \mathcal{A}L) \xrightarrow{\mathbf{d}_\times} \mathbf{C}^{n+1}(G, \mathcal{A}L) \xrightarrow{\mathbf{d}_\times} \mathbf{C}^{n+2}(G, \mathcal{A}L)$$

*vanishes.*

**Proof.** This follows simply from Theorems 11 and 15.  $\square$

## 6. The Comparison between the Multiplicative Coboundary Operator and Kock's Contour Derivatives from Dimension 1 to Dimension 2

Kock's contour derivative  $\mathbf{d}_\circ : \mathbf{C}^0(G, \mathcal{A}L) \rightarrow \mathbf{C}^1(G, \mathcal{A}L)$  should unquestionably agree with the multiplicative coboundary operator  $\mathbf{d}_\times : \mathbf{C}^0(G, \mathcal{A}L) \rightarrow \mathbf{C}^1(G, \mathcal{A}L)$ , for which there is nothing to discuss. Kock's succeeding contour derivative  $\mathbf{d}_\circ : \mathbf{C}^1(G, \mathcal{A}L) \rightarrow \mathbf{C}^2(G, \mathcal{A}L)$  should diverge from  $\mathbf{d}_\times : \mathbf{C}^1(G, \mathcal{A}L) \rightarrow \mathbf{C}^2(G, \mathcal{A}L)$ , and it is our main concern



here how  $\mathbf{d}_\circ : \mathbf{C}^1(G, \mathcal{A}L) \rightarrow \mathbf{C}^2(G, \mathcal{A}L)$  diverges from  $\mathbf{d}_\times : \mathbf{C}^1(G, \mathcal{A}L) \rightarrow \mathbf{C}^2(G, \mathcal{A}L)$  exactly.

Given  $\gamma \in \mathcal{A}^2G$ , we define  $\mathbf{d}_\circ\omega(\gamma) \in \mathcal{A}L$  to be

$$\begin{aligned} & (\mathbf{d}_\circ\omega(\gamma))_{d_1d_2} \\ &= \omega(\gamma_0^1)_{-d_2} \rho_{((\gamma_2)_{d_2})^{-1}} (\omega(\gamma_{d_2}^2)_{-d_1}) \rho_{((\gamma_1)_{d_1})^{-1}} (\omega(\gamma_{d_1}^1)_{d_2}) \omega(\gamma_0^2)_{d_1} \end{aligned} \quad (4)$$

for  $d_1, d_2 \in D$ . Obviously we have to show that

**Proposition 17.** *For any  $\omega \in \mathbf{C}^1(G, \mathcal{A}L)$  and any  $\gamma \in \mathcal{A}^2G$ , there exists a unique  $\mathbf{d}_\circ\omega(\gamma) \in \mathcal{A}L$  abiding by condition 4.*

**Proof.** It suffices to note that if  $d_1 = 0$  or  $d_2 = 0$ , then the right hand side of (4) is equal to the identity at  $\gamma(0, 0)$ , which is easy to see.  $\square$

Now we are ready to establish the main result of Kock [5] in our general context.

**Theorem 18.** *Given  $\omega \in \mathbf{C}^1(G, \mathcal{A}L)$  and  $\gamma \in \mathcal{A}^2G$ , we have*

$$\mathbf{d}_\circ\omega(\gamma) = \mathbf{d}_\times\omega(\gamma) + [\omega(\gamma_0^2), \omega(\gamma_0^1)].$$

**Proof.** Let  $d_1, d_2 \in D$ . Then we have

$$\begin{aligned} & \{\mathbf{d}_\circ\omega(\gamma) - \mathbf{d}_\times\omega(\gamma)\}_{d_1d_2} \\ &= \{\mathbf{d}_\times\omega(\gamma)\}_{-d_1d_2} \{\mathbf{d}_\circ\omega(\gamma)\}_{d_1d_2} \\ &= \{\omega(\gamma_0^2)_{-d_1} \rho_{((\gamma_2)_{d_2})^{-1}} (\omega(\gamma_{d_2}^2)_{d_1}) \rho_{((\gamma_1)_{d_1})^{-1}} (\omega(\gamma_{d_1}^1)_{-d_2}) \omega(\gamma_0^1)_{d_2}\} \\ & \quad \{\omega(\gamma_0^1)_{-d_2} \rho_{((\gamma_2)_{d_2})^{-1}} (\omega(\gamma_{d_2}^2)_{-d_1}) \rho_{((\gamma_1)_{d_1})^{-1}} (\omega(\gamma_{d_1}^1)_{d_2}) \omega(\gamma_0^2)_{d_1}\} \\ &= \omega(\gamma_0^2)_{-d_1} \rho_{((\gamma_2)_{d_2})^{-1}} (\omega(\gamma_{d_2}^2)_{d_1}) \rho_{((\gamma_1)_{d_1})^{-1}} (\omega(\gamma_{d_1}^1)_{-d_2}) \rho_{((\gamma_2)_{d_2})^{-1}} \\ & \quad (\omega(\gamma_{d_2}^2)_{-d_1}) \rho_{((\gamma_1)_{d_1})^{-1}} (\omega(\gamma_{d_1}^1)_{d_2}) \omega(\gamma_0^2)_{d_1}. \end{aligned}$$

Since we have

$$\omega(\gamma_0^2)_{-d_1} \rho_{((\gamma_2)_{d_2})^{-1}}(\omega(\gamma_{d_2}^2)_{d_1}) = (\mathbf{D}F_2)_{d_1 d_2}$$

$$\rho_{((\gamma_1)_{d_1})^{-1}}(\omega(\gamma_{d_1}^1)_{-d_2}) = (\mathbf{D}F_1)_{-d_1 d_2} \omega(\gamma_0^1)_{-d_2}$$

$$\rho_{((\gamma_2)_{d_2})^{-1}}(\omega(\gamma_{d_2}^2)_{-d_1}) = (\mathbf{D}F_2)_{-d_1 d_2} \omega(\gamma_0^1)_{-d_1}$$

$$\rho_{((\gamma_1)_{d_1})^{-1}}(\omega(\gamma_{d_1}^1)_{d_2}) = (\mathbf{D}F_1)_{d_1 d_2} \omega(\gamma_0^1)_{d_2}$$

we can continue our calculation as follows:

$$\begin{aligned} & \{\mathbf{d}_\circ \omega(\gamma) - \mathbf{d}_\times \omega(\gamma)\}_{d_1 d_2} \\ &= (\mathbf{D}F_2)_{d_1 d_2} (\mathbf{D}F_1)_{-d_1 d_2} \omega(\gamma_0^1)_{-d_2} (\mathbf{D}F_2)_{-d_1 d_2} \omega(\gamma_0^1)_{-d_1} (\mathbf{D}F_1)_{d_1 d_2} \omega(\gamma_0^1)_{d_2} \omega(\gamma_0^2)_{d_1}. \end{aligned}$$

It is easy to see that  $(\mathbf{D}F_1)_{d_1 d_2}$ ,  $(\mathbf{D}F_1)_{-d_1 d_2}$ ,  $(\mathbf{D}F_2)_{d_1 d_2}$  and  $(\mathbf{D}F_2)_{-d_1 d_2}$  commute with any term occurring in the above calculation, so that the calculation itself can step forward as follows with  $(\mathbf{D}F_1)_{d_1 d_2}$  and  $(\mathbf{D}F_2)_{d_1 d_2}$  canceling out  $(\mathbf{D}F_1)_{-d_1 d_2}$  and  $(\mathbf{D}F_2)_{-d_1 d_2}$  respectively:

$$\begin{aligned} & \{\mathbf{d}_\circ \omega(\gamma) - \mathbf{d}_\times \omega(\gamma)\}_{d_1 d_2} \\ &= \omega(\gamma_0^1)_{-d_2} \omega(\gamma_0^1)_{-d_1} \omega(\gamma_0^1)_{d_2} \omega(\gamma_0^2)_{d_1} \\ &= [\omega(\gamma_0^2), \omega(\gamma_0^1)]_{d_1 d_2}. \end{aligned}$$

Since  $d_1 \in D$  and  $d_2 \in D$  were arbitrary, the desired conclusion follows at once.  $\square$

**Corollary 19.** *For any  $\omega \in \mathbf{C}^1(G, AL)$ , we have  $\mathbf{d}_\circ \omega \in \mathbf{C}^2(G, AL)$ .*

**Corollary 20.** *If  $\omega \in \mathbf{C}^1(G, AL)$  is a closed form (i.e.,  $\mathbf{d}_\times \omega = 0$ ), then we have*

$$\mathbf{d}_\circ \omega(\gamma) = [\omega(\gamma_0^2), \omega(\gamma_0^1)].$$

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